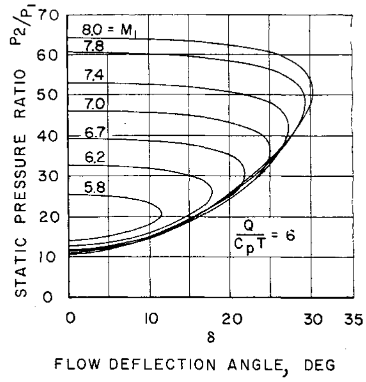

Fig. 4

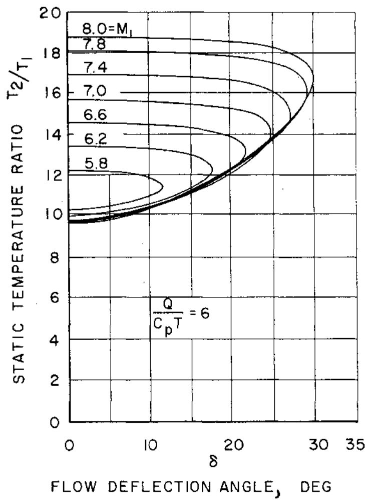

Fig. 5

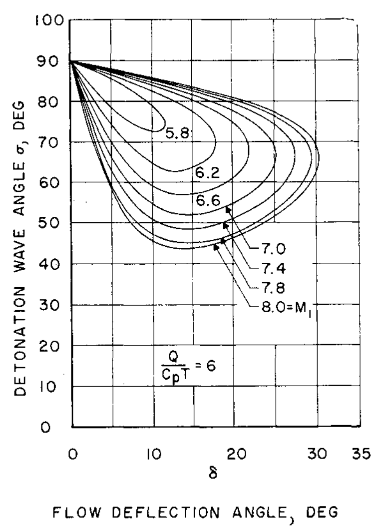

Fig. 6

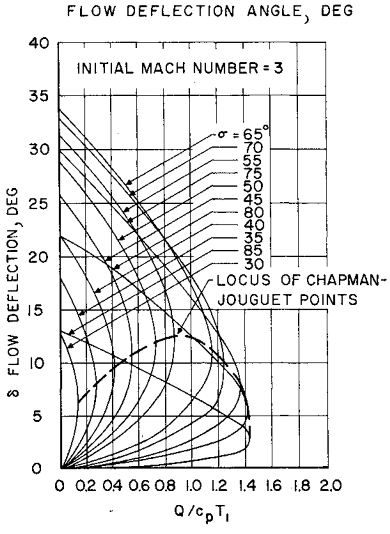

Fig. 7

$$M_2^2 \sin^2(\sigma - \delta) = 1 \quad (16)$$

or

$$M_{2n} = 1 \quad (16a)$$

Results

Some typical plots of the solutions of these oblique detonation wave equations are shown in Figs. 2–7. As with normal detonation waves, for a given value of Q/c_pT_1, there is a minimum value of M_1, the Chapman-Jouguet condition, which permits real values for the radical in Eq. (6). Below this minimum, the assumption of steady flow is violated. Figures 2–6 represent Eqs. (6–10). Figure 7 represents the wave behavior at a fixed Mach number, $M_1 = 3$, as the heat addition parameter (sometimes called Damkohler second parameter) is varied. The locus of the Chapman-Jouguet points is shown. The region above the *C-J* line in Fig. 7 represents the strong detonation solutions.

References

[1] Gross, R. A. and Chinitz, W., "A study of supersonic combustion," J. Aerospace Sci. **27**, 517–525 (1960).

[2] Rhodes, R. P., Rubins, P. M., and Chriss, D. E., "The effect of heat release on the flow parameters in shock-induced combustion," Arnold Eng. Dev. Center Tech. Doc. Rept. AEDC-TOR-62-78 (May 1962).

[3] Samaras, D. G., "Gas dynamic treatment of exothermic and endothermic discontinuities," Can. J. Res. **26**, 1–21 (1948).

[4] Siestrunck, R., Fabri, J., and Le Grives, E., "Some properties of stationary detonation waves," *Fourth Symposium on Combustion* (Williams and Wilkins Co., Baltimore, Md., 1953), pp. 448–501.

[5] Rutkowski, J. and Nicholls, J. A., "Considerations for the attainment of a standing detonation wave," *Proceedings of the Gas Dynamics Symposium on Aerothermochemistry, Northwestern University, August 1955* (Northwestern University Press, Evanston, Ill., 1955), pp. 243–253.

Forced-Convection Heat Transfers with Time-Dependent Surface Temperatures

Eldon L. Knuth*

University of California, Los Angeles, Calif.

Aerothermodynamicists frequently calculate the rate of forced-convection heat transfer to a surface with time-dependent temperature, e.g., to the external surface of a high-speed vehicle subjected to thermal radiation from a nuclear explosion. The applicability of the equations developed for steady-state forced-convection heat transfer has been studied independently and simultaneously by Knuth and Bussell[1] and by Sparrow and Gregg;[2] results of a related study have been published recently by Goodman.[3] In the first paper, the geometrical aspects of the problem are simplified by considering Couette flow; initial conditions are specified, and heat transfer rates for subsequent values of time are computed. In the second paper, the temporal aspects are simplified by neglecting the past history of the system and examining the departure of the instantaneous conditions from quasi-steady conditions; computations are made for laminar boundary layer flow of a compressible fluid with Prandtl number of 0.72. In the third paper, the geometrical aspects are simplified by using an integral method; equations are developed for laminar boundary layer flow of an incompressible fluid with arbitrary Prandtl number. The purpose of the present note is to show how these three studies com-

Received January 25, 1963.

* Associate Professor and Head of Molecular Beam Laboratory, Department of Engineering. Associate Fellow Member AIAA.

plement each other and to suggest a useful combination of the results of these studies.

In Ref. 1, for high-speed laminar Couette flow of a fluid with constant properties, the temperature distribution is described by

$$\partial\theta/\partial\tau = \partial^2\theta/\partial\eta^2 \quad (1)$$

where

$$\theta \equiv \frac{T\{1 + Pr[(\gamma - 1)/2]M^2\} - T_r}{T_w(0) - T_r}$$

$$\tau \equiv \alpha t/\delta^2 \qquad \eta \equiv y/\delta$$

where T is temperature (time-dependent), Pr Prandtl number, γ specific heat ratio, M Mach number, α thermal diffusivity, t time, δ distance between the two surfaces, y distance from the fixed surface, and subscripts r and w refer to recovery and wall conditions, respectively. Boundary and initial conditions for the case in which quasi-steady conditions exist at $t = 0$ are given by

$$\begin{array}{lll} \eta = 1 & \tau > 0 & \theta = 0 \\ \eta = 0 & \tau > 0 & \theta = f(\tau) \\ 0 \leqslant \eta \leqslant 1 & \tau = 0 & \theta = 1 - \eta \end{array} \quad (2)$$

Carslaw and Jaeger (see Ref. 4, p. 86) give the solution

$$\theta = 2\sum_{n=1}^{\infty} e^{-n^2\pi^2\tau} \sin n\pi\eta \times \left[\int_0^1 (1-\eta)\sin n\pi\eta\, d\eta + n\pi\int_0^\tau e^{n^2\pi^2\tau'} f(\tau')d\tau'\right] \quad (3)$$

Integrating the first integral directly and the second integral by parts,

$$\theta = 2\sum_{n=1}^{\infty} \frac{\sin n\pi\eta}{n\pi}\left[f(\tau) - e^{-n^2\pi^2\tau}\int^\tau e^{n^2\pi^2\tau'}\frac{df(\tau')}{d\tau'}d\tau'\right] \quad (4)$$

The series

$$2\sum_{n=1}^{\infty}\frac{\sin n\pi\eta}{n\pi}$$

is the Fourier sine series for the function $1 - \eta$. Hence,

$$\theta = (1-\eta)f(\tau) - 2\sum_{n=1}^{\infty}\frac{\sin n\pi\eta}{n\pi}e^{-n^2\pi^2\tau} \times \int_0^\tau e^{n^2\pi^2\tau'}\frac{df(\tau')}{d\tau'}d\tau' \quad (5)$$

The instantaneous heat transfer rate at the wall is given by

$$q_{\text{inst}} = -\frac{k}{\delta}[T_w(0) - T_r]\left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=0}$$

$$= \frac{k}{\delta}[T_w(\tau) - T_r] + \frac{k}{\delta}[T_w(0) - T_r]\, 2\sum_{n=1}^{\infty}e^{-n^2\pi^2\tau} \times \int_0^\tau e^{n^2\pi^2\tau'}\frac{df(\tau')}{d\tau'}d\tau'$$

$$= q_{\text{quasi-steady}}\left[1 + \frac{2}{f(\tau)}\sum_{n=1}^{\infty}e^{-n^2\pi^2\tau} \times \int_0^\tau e^{n^2\pi^2\tau'}\frac{df(\tau')}{d\tau'}d\tau'\right] \quad (6)$$

The first term is a quasi-steady heat transfer rate, whereas the remainder is the deviation from the quasi-steady rate caused by the nonzero heat capacity of the fluid. If the magnitude of the remainder is small in comparison with the magnitude of the first term, then one may calculate the heat transfer rate to good approximation using equations developed for steady-state systems. Note that the remainder includes effects of the past history of the system.

In order to facilitate comparing Eq. (6) with the results of Refs. 2 and 3, integrate by parts twice to obtain

$$q_{\text{inst}} = q_{\text{quasi-steady}}\left[1 + \frac{1}{3}\frac{f'(t)}{f(t)}\left(\frac{\delta^2}{\alpha}\right) - \frac{1}{45}\frac{f''(t)}{f(t)}\left(\frac{\delta^2}{\alpha}\right)^2 - \frac{1}{3}\frac{z_2(\tau)}{z_2(0)}\frac{f'(0)}{f(t)}\left(\frac{\delta^2}{\alpha}\right) + \frac{1}{45}\frac{z_4(\tau)}{z_4(0)}\frac{f''(0)}{f(t)}\left(\frac{\delta^2}{\alpha}\right)^2 + \dots\right] \quad (7)$$

where

$$z_s(\tau) \equiv 2\sum_{n=1}^{\infty}\frac{e^{-n^2\pi^2\tau}}{(n\pi)^s}$$

$$z_s(0) = 2\zeta(s)/\pi^s$$

and where $\zeta(s)$ is the Riemann zeta function, $z_2(0) = \frac{1}{3}$, and $z_4(0) = \frac{1}{45}$. [Variations of the ratios $z_2(\tau)/z_2(0)$ and $z_4(\tau)/z_4(0)$ with dimensionless time τ are shown in Fig. 1.] This equation may be compared directly with Eq. (15) of Ref. 2:

$$q_{\text{inst}} = q_{\text{quasi-steady}}\left[1 + 2.39\frac{f'(t)}{f(t)}\left(\frac{x}{u_\infty}\right) - 0.801\frac{f''(t)}{f(t)}\left(\frac{x}{u_\infty}\right)^2 + \dots\right] \quad (8)$$

where x is distance from the leading edge and u_∞ is freestream velocity. (Recall that this equation was derived neglecting the past history of the system and setting $Pr = 0.72$.) A third version may be obtained if one begins with Eq. (36) of Ref. 3; replacing the term corresponding to a temperature jump at $t = 0$ by a term corresponding to quasi-steady heat transfer at $t = 0$, one obtains

$$q_{\text{inst}} = q(0)\left[1 + \int_0^F \frac{1}{\phi(F - F')}\frac{df(F')}{dF'}dF'\right] \quad (9)$$

where $F \equiv 0.20\, Pr^{-1/3}\, u_\infty t/x$, and $\phi(F)$ is displayed graphically in Fig. 1 of Ref. 3. (Recall that this equation was derived using an integral method.) Approximating $\phi(F)$ by $\phi = (2F)^{1/2}$ for $F < F^*$ and by $\phi = 1$ for $F > F^* \equiv 0.266$, integrating by parts, and using Taylor's series, one obtains

$$q_{\text{inst}} = q_{\text{quasi-steady}}\left[1 + 2.32\, Pr^{1/3}\frac{f'(t)}{f(t)}\left(\frac{x}{u_\infty}\right) - 0.732\, Pr^{2/3}\frac{f''(t)}{f(t)}\left(\frac{x}{u_\infty}\right)^2 - 2.32\, C_2(F)Pr^{1/3}\frac{f'(0)}{f(t)}\left(\frac{x}{u_\infty}\right) + 0.732\, C_4(F)Pr^{2/3}\frac{f''(0)}{f(t)}\left(\frac{x}{u_\infty}\right)^2 + \dots\right] \quad (10)$$

where

$$C_2(F) \equiv 1 - \frac{(2F)^{1/2} - F}{(2F^*)^{1/2} - F^*} \qquad F < F^*$$

$$\equiv 0 \qquad F > F^*$$

and

$$C_4(F) \equiv 1 - \frac{(2F)^{3/2} - 3F^2}{(2F^*)^{3/2} - 3F^{*2}} - 15.8FC_2(F) \qquad F < F^*$$

$$\equiv 0 \qquad F > F^*$$

[Variations of the factors $C_2(F)$ and $C_4(F)$ with dimensionless time F also are shown in Fig. 1.] The dimensionless time F plays apparently the same role in boundary layer flows as does the dimensionless time τ in Couette flows; values of

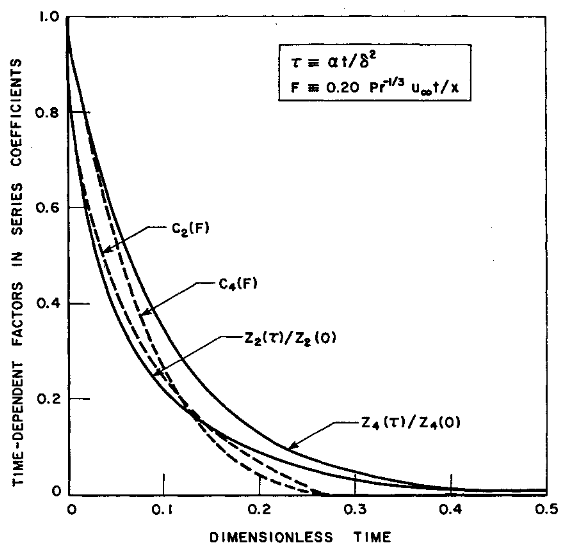

Fig. 1 Variations of the factors $z_2(\tau)/z_2(0)$, $z_4(\tau)/z_4(0)$, $C_2(F)$, and $C_4(F)$ with dimensionless times τ and F

the factors $C_2(F)$ and $C_4(F)$, obtained for boundary layer flows using an integral method, are seen to differ only slightly from values of the factors $z_2(\tau)/z_2(0)$ and $z_4(\tau)/z_4(0)$ obtained for Couette flows using an exact method. Note that, if the time elapsed since quasi-steady conditions existed is small, i.e., if $F \ll 1$ and $\tau \ll 1$, then the terms corresponding to deviations from the quasi-steady heat transfer rate in Eqs. (7) and (10) disappear, whereas the corresponding terms in Eq. (8) do not. This situation is in keeping with the fact that Refs. 1 and 3 take into account the past history of the system, whereas Ref. 2 does not.

An expression more useful than Eqs. (7, 8, and 10) would be obtained if one could combine the most useful features of these equations into a single expression. One might combine those features of Eq. (7) which are unique to an exact treatment of the past history of the system, those features of Eq. (8) which are unique to an exact treatment of the two-dimensional nature of compressible boundary layer flows, and those features of Eq. (10) which are unique to a treatment in which the value of the Prandtl number is arbitrary, and write

$$q_{\text{inst}} = q_{\text{quasi-steady}} \left[1 + 2.67\ Pr^{1/3} \frac{f'(t)}{f(t)} \left(\frac{x}{u_\infty}\right) - 1.00\ Pr^{2/3} \frac{f''(t)}{f(t)} \left(\frac{x}{u_\infty}\right)^2 - 2.67\ \frac{z_2(F)}{z_2(0)}\ Pr^{1/3} \frac{f'(0)}{f(t)} \left(\frac{x}{u_\infty}\right) + 1.00\ \frac{z_4(F)}{z_4(0)}\ Pr^{2/3} \frac{f''(0)}{f(t)} \left(\frac{x}{u_\infty}\right)^2 + \ldots \right] \quad (11)$$

If $Pr^{1/3}\, x/u_\infty$ is replaced by $\delta^2/8\alpha$, then Eq. (7) is retrieved to fair approximation; if $F \gg 1$ and $Pr = 0.72$, then Eq. (8) is retrieved exactly, whereas, if $z_2(F)/z_2(0)$ and $z_4(F)/z_4(0)$ are replaced by $C_2(F)$ and $C_4(F)$, respectively, then Eq. (10) is retrieved to fair approximation.

Fortunately, and as indicated in Refs. 1 and 2, the aerothermodynamicist will find that most forced-convection heat transfer rates to surfaces with time-dependent temperatures may be calculated to good approximation using equations developed for the steady-state case.

References

[1] Knuth, E. L. and Bussell, B., "Analytical studies of aircraft structures exposed to transient external heating, Vol. IV: Heat-transfer coefficients for speeds up to $M = 3$, with emphasis on effects of wall-temperature variations," Wright Air Dev. Center TR 54-579 (AD 131015), Dept. Eng., Univ. Calif., Los Angeles (September 1957).

[2] Sparrow, E. M. and Gregg, J. L., "Nonsteady surface temperature effects on forced convection heat transfer," J. Aerospace Sci. **24**, 776–777 (1957).

[3] Goodman, T. R., "Effect of nonsteady wall temperature on incompressible heat transfer," J. Heat Transfer, Trans. Am. Soc. Mech. Engrs. **C84**, 347–352 (1962).

[4] Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids* (Oxford University Press, London, 1947).

Orbital Transfer in Minimum Time

W. E. Bleick*

U. S. Naval Postgraduate School, Monterey, Calif.

The problem of orbital transfer discussed here is that of scheduling the direction p of constant momentum thrust of a rocket, which loses mass at a constant rate, so that it transfers to a known earth satellite orbit in a minimum time T after launching. The launching conditions are assumed to be fixed. Figure 1 illustrates the problem for a circular orbit. To aid the discussion, imaginary physical rendezvous of the rocket and a target satellite is assumed to occur at the transfer sector angle B. The calculus of variations problem is set up and solved numerically by the method of Faulkner.[1] The nonrotating Oxy axes shown in Fig. 1 are used. The coordinates and velocity components of the rocket and target satellite are denoted by x,y,u,v and X,Y,U,V, respectively. For simplicity the equations of motion of rocket and target will be written in a nondimensional form by using suitable units. The unit of length is taken as the earth's equatorial radius, $R_e = 20{,}925{,}000$ ft. The unit taken for time t is the time required by a hypothetical earth satellite, in equatorial, circular, vacuum, sea-level orbit, to traverse a sector of 1 rad. This unit of time is $(R_e/g)^{1/2} = 13.459$ min, where $g = 32.086$ ft/sec^2 is the acceleration of gravity at the equator. The unit of velocity is then the speed of this hypothetical satellite. These units of length and time always will be understood, unless other units are mentioned specifically.

Statement of the Problem

The equations of motion of the rocket are

$$\varphi_1 = \dot{u} - g_1 - a \cos p = 0 \qquad \varphi_3 = \dot{x} - u = 0$$
$$\varphi_2 = \dot{v} - g_2 - a \sin p = 0 \qquad \varphi_4 = \dot{y} - v = 0 \quad (1)$$

where $g_1 = -x/r^3$, $g_2 = -y/r^3$, $r^2 = x^2 + y^2$, and $a = c\dot{M}/(1 - \dot{M}t)g$, where $\dot{M}$ is the constant fraction of initial gross rocket mass lost per unit time, and c is the constant speed of the emitted rocket gases. The fixed launching conditions are taken as

$$x(0) = 0 \qquad u(0) = V_1 = V_0 \cos\theta$$
$$y(0) = 1 \qquad v(0) = V_2 = V_0 \sin\theta \quad (2)$$

The terminal point of the rocket trajectory is variable with

$$x(T) = X(T) \qquad y(T) = Y(T)$$
$$u(T) = U(T) \qquad v(T) = V(T) \quad (3)$$

Note that Eqs. (1) and (3) imply that $\dot{U}(T) = -[X/R^3]_T = g_1(T)$ and $\dot{V}(T) = -[Y/R^3]_T = g_2(T)$, where $R^2 = X^2 + Y^2$. It is assumed also that the rocket thrust is turned off abruptly at time T. The problem is to choose the control variable p to

Received January 30, 1963. This work was supported by the Office of Naval Research. The author is indebted to his colleague Frank D. Faulkner for suggesting the problem of this paper and for discussions of difficulties encountered.

* Professor, Department of Mathematics and Mechanics.